Optimal quantum communication using multiparticle partially entangled states

Atul Kumar,[1,*] Satyabrata Adhikari,[1,†] Subhashish Banerjee,[1,‡] and Sovik Roy[2,§]

[1]Indian Institute of Technology Rajasthan, Jodhpur, india

[2]Techno India, Salt Lake City, Kolkata 700091, India

(Dated: January 16, 2013)

We propose a three-qubit partially entangled set of states as a shared resource for optimal and faithful quantum information processing. We show that our states always violate the Svetlichny inequality, which is a Bell type inequality whose violation is a sufficient condition for the confirmation of genuine three-qubit nonlocality. Although, our states can be physically realized from the generalized Greenberger-Horne-Zeilinger (GGHZ) states using a simple quantum circuit, the non-local properties of the set are quite different from the generalized GHZ states but are similar to the maximal slice states (MS); even though our states are not locally equivalent to the MS states. Unlike other two and three-qubit partially entangled states, quantum teleportation using our states results in faithful transmission of information with unit probability and unit fidelity by performing only standard measurements for the sender, controller and receiver. We further demonstrate that dense coding also leads to the deterministic transfer of maximum number of bits from the sender to the receiver. We also introduce witness operators able to experimentally detect the family of states introduced. This work highlights the importance of both the local as well as non-local aspects of quantum correlations in multi-qubit systems.

PACS numbers: 03.67.Hk, 03.67.-a

I. INTRODUCTION

Quantum entanglement is efficiently used as a key resource in several communication protocols for sending quantum as well as classical information from a sender to a receiver [1–4]. For communication tasks including only two parties, Bell states play a pivotal role in transferring the information from one location to another arbitrary location [1, 4]. Quantum information transfer protocols with more than two parties allow for controlled quantum communication where the controller controls or assists the successful information transfer between the sender and the receiver [5–9]. In general, for an optimal and successful information transfer, the shared quantum channel between two or more than two parties is considered to be a maximally entangled resource. However, in real experimental set-ups, it is always a challenge to obtain a multiqubit maximally entangled resource [10]. Therefore, it is important to identify multiqubit entangled systems which are partially entangled but can be efficiently used as a resource in quantum information processing with optimal success.

In this article, we address this issue by proposing a set of three-qubit partially entangled states as a resource for faithful quantum information processing protocols and demonstrate their successful applications for controlled quantum teleportation, controlled quantum secure deterministic communication and controlled dense coding. We show that our states can be prepared from a three-qubit partially entangled GHZ state [11] using a simple quantum circuit, but surprisingly outperforms the partially entangled GHZ state as an entangled resource for various quantum information processing protocols. For the dense coding protocol, unlike the partially entangled GHZ or the MS [12] states, our states allow the transfer of maximum number of bits from the sender to the receiver, for a general measurement performed by the controller. The use of partially entangled two-qubit Bell and three-qubit GHZ or W states [13] leads to the dependence of probability and fidelity on the parameters of the unknown state to be teleported and the process is not faithful. In all such cases one can teleport a single qubit using partially entangled states as a quantum channel through conclusive teleportation, qubit assisted conclusive teleportation and by performing projections in a non maximally entangled basis, but with less than unit probability [14–16]. Dealing with three-qubit pure entangled states, it was shown that there exist a class of three qubit non-maximally entangled W states which are useful for deterministic teleportation with unit fidelity [17]. But in this protocol one needs to carry out three-qubit von Neumann measurements, which in general is a difficult task experimentally. In contrast to this, the partially entangled states proposed by us require only two-qubit and single-qubit measurements, an experimentally easier task.

Thus, the set of states proposed in this article provides a flexible approach to quantum information processing by ensuring the success of protocols within the constraints of the existing experimental set-ups for multiqubit states. The advantages are that the set can be prepared using a simple quantum circuit and also releases the constraints of using maximally entangled states for communication tasks, making them interesting candidates for practical

*atulk@iitj.ac.in
†satya@iitj.ac.in
‡subhashish@iitj.ac.in
§sovikr@rediffmail.com

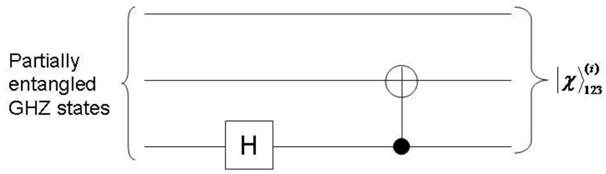

FIG. 1: Quantum circuit to prepare the $|\chi\rangle^{(i)}_{123}$ states.

implementation of quantum information processing protocols using cavity quantum electrodynamics, optical or atomic systems.

The plan of the paper is as follows. In the next section we introduce our set of states and demonstrate a simple quantum circuit that can be used to build it up. We then compute its tangle and compare its genuine three-partite entanglement properties using the Svetlichny inequality, with the maximal sliced states. Optimal witness operators detecting our states are then constructed. We then make applications of our states to dense coding and teleportation. Interestingly, even though our states are nonlocally equivalent to the maximal slice (MS) states, they are locally inequivalent, leading to their enhanced efficiency in dense coding. This highlights the need for an understanding of both the local and non-local facets of quantum correlations in multi-qubit systems.

II. THE PARTIALLY ENTANGLED QUANTUM CHANNEL AND ITS PROPERTIES

The three-qubit partially entangled set of states that we characterize in this article can be represented as

$$\begin{aligned} |\chi\rangle^{(1),(2)}_{123} &= \frac{1}{\sqrt{2}}\left[\sin\theta\,|000\rangle_{123} \pm \sin\theta\,|011\rangle_{123}\right. \\ &+ \left.\cos\theta\,|110\rangle_{123} \mp \cos\theta\,|101\rangle_{123}\right]. \quad (1) \end{aligned}$$

The meaning of $\sin(\theta)$, $\cos(\theta)$ becomes clear from the simple quantum circuit used to illustrate the preparation of the orthonormal entangled set of states $|\chi\rangle^{(i)}_{123}$, as shown in Fig. 1. The input in the circuit is the generalized GHZ states. For example, if the input in the circuit is

$$|\psi\rangle^{(1)}_{GHZ} = \sin\theta\,|000\rangle_{123} + \cos\theta\,|111\rangle_{123}\,, \quad (2)$$

then applying a Hadamard operation on the 3^{rd} qubit will result in

$$\begin{aligned} |\psi'\rangle^{(1)} &= \frac{1}{\sqrt{2}}\left[\sin\theta\,|000\rangle_{123} + \sin\theta\,|001\rangle_{123} + \cos\theta\,|110\rangle_{123}\right. \\ &- \left.\cos\theta\,|111\rangle_{123}\right]. \quad (3) \end{aligned}$$

$|\psi'\rangle^{(1)}$ can be transformed to the three-qubit partially entangled state $|\chi\rangle^{(1)}_{123}$ by performing a C-NOT operation on qubits 2 and 3 with qubit 3 as control and qubit 2 as target. Similarly, one can obtain other orthogonal states of the set $|\chi\rangle^{(i)}_{123}$, depending on the input given to the circuit. Thus, the partially entangled set represented in Eq. (1) can be obtained starting from the generalized GHZ states by performing one local and one non-local operation. As we will demonstrate in the following sections, this additional non-local C-NOT operation leads to an efficient and optimally controlled quantum communication when the $|\chi\rangle^{(i)}_{123}$ states are used as a quantum channel compared to other partially entangled sates.

In order to analyze and compare the entanglement properties of the partially entangled state $|\chi\rangle^{(i)}_{123}$ with other three-qubit partially entangled states such as the partially entangled GHZ state (GGHZ) in Eq. (2) and the maximal slice (MS) state(s) [12], given by

$$|\phi\rangle_{MS} = \frac{1}{\sqrt{2}}\left[|000\rangle + \cos\theta_1\,|110\rangle + \sin\theta_1\,|111\rangle\right]_{123}, \quad (4)$$

we use the 3-tangle (τ) [18] as a measure for genuine tri-partite entanglement;

$$\tau = C^2_{1(23)} - C^2_{12} - C^2_{13}, \quad (5)$$

where $C^2_{1(23)}$ is a measure of the entanglement between qubit 1 and the joint state of qubits 2 and 3 and C^2_{12} (or C^2_{13}) measures the entanglement between qubits 1 and 2 (or 3). Note that for $\theta_1 = \pi/2$, the MS state is exactly the same as the three-qubit GHZ state and for $\theta = \pi/4$, the $|\chi\rangle_{123}$ state is locally equivalent to the GHZ state. The 3-tangle for the state $|\chi\rangle_{123}$ is $\sin^2 2\theta$ which varies from 0 for a product state to 1 for a maximum entanglement state. For example, the value of τ is 0 ($\theta = 0$) for the state $|1\rangle_1 \otimes \frac{1}{\sqrt{2}}\left[|10\rangle_{23} - |01\rangle_{23}\right]$ which is a biseparable state and 1 ($\theta = \pi/4$) for the state $\frac{1}{2}\left[|000\rangle_{123} + |011\rangle_{123} + |110\rangle_{123} - |101\rangle_{123}\right]$, which is a maximally entangled state. Moreover, in comparison to the $|\chi\rangle_{123}$ state, the value of the 3-tangle for the GGHZ and MS states are $\sin^2 2\theta$ and $\sin^2\theta_1$, respectively. Thus, all the three partially entangled states show genuine but less than maximum entanglement between the three qubits for $0 < \theta < \pi/4$ and $0 < \theta_1 < \pi/2$.

We also characterize the entanglement between three qubits in the GGHZ, MS and $|\chi\rangle_{123}$ states using the Svetlichny inequality [19, 20] whose violation is sufficient for the confirmation of genuine tri-partite nonlocal correlations. In order to distinguish between two-qubit versus three-qubit nonlocality, Svetlichny considered a hybrid model where if nonlocal correlations can exist between any two of the qubits, then the Svetlichny's inequality holds

$$S(|\psi\rangle) = \left|\left\langle \begin{array}{l} A(BC + BC' + B'C - B'C') \\ +A'(BC - BC' - B'C - B'C') \end{array}\right\rangle\right| \leq 4. \quad (6)$$

Here the measurements $A = \vec{a}.\vec{\sigma}_1$ or $A' = \vec{a}'.\vec{\sigma}_1$ are performed on qubit 1 and $\vec{a}$, $\vec{a}'$ are unit vectors. The measurements B or B' and C or C' are defined in a similar fashion for the qubits 2 and 3, respectively, and σ is a spin projection operator. A plot of the maximum ex-

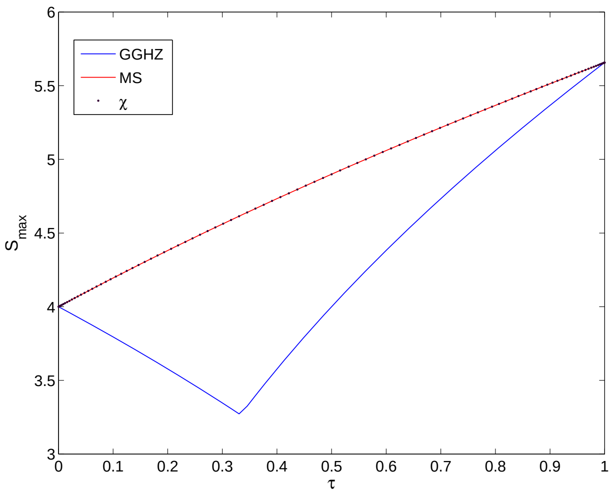

FIG. 2: Comparison of numerical calculation for the value of $S(|\psi\rangle)$ versus τ for generalized GHZ (GGHZ) states (solid line, blue), maximal slice state (solid line, red) and $|\chi\rangle_{123}^{(1)}$ state (dots, violet).

pectation value of the Svetlichny operator versus the 3-tangle (τ) for the GGHZ, MS and $|\chi\rangle_{123}$ states shown in Fig. (2), clearly illustrating the differences and similarities between the nonlocal properties of these states. The GGHZ states with $\tau \leq 1/2$ do not violate the Svetlichny inequality but it is always violated by all the MS and $|\chi\rangle_{123}^{(i)}$ states. The nonlocal correlations in the $|\chi\rangle_{123}$ state are different from the nonlocal correlations in the GGHZ states but appear to be exactly the same as in the MS states. This raises the question whether there exists any local unitary transformation such that the MS states can be locally transformed to the $|\chi\rangle_{123}$ states? However following [21], we found that these two states are not local unitary equivalents of each other.

The difference in the nonlocal correlations between the GGHZ states and the partially entangled $|\chi\rangle_{123}^{(i)}$ states leads to the efficient use of the latter as an entangled resource in different quantum information processing tasks.

Based on our discussions about the physical realization and entanglement properties of the $|\chi\rangle_{123}^{(i)}$ states above, we now proceed to demonstrate the usefulness of such states as a shared resource for controlled quantum information processing.

III. WITNESS OPERATOR

The experimental detection of entanglement is facilitated by the existence of entanglement witnesses [22, 23] which are Hermitian operators with at least one negative eigenvalue. For example, witness operators can be decomposed in terms of Pauli spin matrices (for qubits) and Gell-Mann matrices (for qutrits and other higher dimensional entities) which are experimentally measurable quantities. In [22] the three-qubit mixed entanglement in different classes were characterized using witness operators. An example of such an operator would be

$$W_{GHZ} = \frac{3}{4}I - P_{GHZ}, \tag{7}$$

where P_{GHZ}=$|GHZ\rangle\langle GHZ|$ and $|GHZ\rangle = \frac{1}{\sqrt{2}}[|000\rangle + |111\rangle]$. Although this witness operator is useful to distinguish three qubit states in GHZ or W class, it cannot detect whether the state $|\chi\rangle_{123}^{(i)}$, given in Eq. (1), can be in the GHZ or W class. This motivates us to formulate a new witness operator for the detection of partially entangled three-qubit states $|\chi\rangle_{123}^{(i)}$. The form of our witness operator is

$$W_\chi = \epsilon I - P_\chi, \tag{8}$$

where $P_\chi = |\chi\rangle\langle\chi|$ and $|\chi\rangle = \frac{1}{\sqrt{2}}[|000\rangle + |011\rangle + |110\rangle - |101\rangle]$. For example, in order to optimize the overlap ϵ between the $|\chi\rangle_{123}^{(1)}$ and $|W\rangle_{123}$ state, we consider

$$|W\rangle_{123} = r_1|000\rangle + r_2[|001\rangle + |010\rangle + |100\rangle]. \tag{9}$$

Following the method developed in [22], we find that the form of our witness operator is

$$W_\chi = \frac{1}{4}I - P_\chi. \tag{10}$$

It is easy to see that $\mathrm{Tr}\,(\mathrm{W}_\chi \mathrm{P}_{\mathrm{W}_{123}}) \geq 0$ for all the W_{123} states, Eq. (9), and

$$\begin{aligned}\mathrm{Tr}\left(\mathrm{W}_\chi \mathrm{P}_{\chi_{123}^{(1)}}\right) &= -\frac{1}{4}\sin^2\theta - \frac{1}{4}\cos^2\theta - \sin\theta\cos\theta \\ &= -(1 + 2\sin 2\theta) < 0. \end{aligned} \tag{11}$$

Thus, our witness operator will detect all the partially entangled three-qubit states between $0 \leq \theta \leq \frac{\pi}{4}$. As our witness operator detects all the states in the given family, it is also an optimal witness operator. Moreover, the witness operator W_χ can be decomposed in the form of Pauli spin matrices as

$$\begin{aligned} W_\chi &= \frac{1}{8}[I\otimes I\otimes I + \sigma_y\otimes\sigma_y\otimes I - \sigma_y\otimes I\otimes\sigma_y \\ &+ I\otimes\sigma_y\otimes\sigma_y - \sigma_z\otimes\sigma_x\otimes\sigma_x + \sigma_x\otimes\sigma_z\otimes\sigma_x \\ &- \sigma_x\otimes\sigma_x\otimes\sigma_z - \sigma_z\otimes\sigma_z\otimes\sigma_z]. \end{aligned} \tag{12}$$

Hence, the experimental detection of entanglement in the $|\chi\rangle_{123}^{(1)}$ state would be facilitated by the existence of the witness operator W_χ. Interestingly, W_χ detects all the states in the $|\chi\rangle_{123}^{(1)}$ family and shows that the states belong to the GHZ class. In a similar way, the optimal witness operator for the state $|\chi\rangle_{123}^{(2)}$ can be constructed.

IV. DENSE CODING

Quantum dense coding deals with efficient information transfer from a sender to a receiver utilizing an entangled channel between the two [1]. We demonstrate an optimal controlled dense coding protocol using any of the partially entangled states $|\chi\rangle_{123}^{(i)}$ shared between Alice (2), Charlie (1) and Bob (3). For example, if Alice wants to send messages to Bob with Charlie acting as a controller, then the shared quantum channel $|\chi\rangle_{123}^{(1)}$ can be rewritten in terms of Charlie's measurement basis ($|x_+\rangle_1 = \cos\alpha|0\rangle_1 + \sin\alpha|1\rangle_1$, $|x_-\rangle_1 = \sin\alpha|0\rangle_1 - \cos\alpha|1\rangle_1$) as

$$\begin{aligned} |\chi\rangle_{123}^{(1)} &= \frac{|x_+\rangle_1}{\sqrt{2}}\left[\sin\theta\cos\alpha\,|00\rangle + \sin\theta\cos\alpha\,|11\rangle\right. \\ &+ \left.\cos\theta\sin\alpha\,|10\rangle - \cos\theta\sin\alpha\,|01\rangle\right] \\ &+ \frac{|x_-\rangle_1}{\sqrt{2}}\left[\sin\theta\cos\alpha\,|00\rangle + \sin\theta\cos\alpha\,|11\rangle\right. \\ &- \left.\cos\theta\sin\alpha\,|10\rangle + \cos\theta\sin\alpha\,|01\rangle\right]. \end{aligned} \tag{13}$$

If Charlie's measurement outcome is $|x_+\rangle_1$, then the rest of the qubits will be projected onto the state

$$\begin{aligned} |\phi\rangle_{23} &= \left[\sin\theta\cos\alpha\,|00\rangle + \sin\theta\cos\alpha\,|11\rangle\right. \\ &+ \left.\cos\theta\sin\alpha\,|10\rangle - \cos\theta\sin\alpha\,|01\rangle\right]. \end{aligned} \tag{14}$$

The dense coding capacity [24] for the partially entangled states, Eq. (14), shared between Alice and Bob, is independent of θ (parameter of shared entangled states) and α (characterizing the controllers measurement basis) and is always equal to two. This holds for all values of θ except zero and $\pi/2$. In order to communicate the required message, Alice first encodes her information using one of the four unitary operators, *for e.g.*, the identity operator I^2 or the Pauli spin operators (σ_z^2, σ_x^2, $i\sigma_y^2$), on her qubit 2 and then sends it to Bob. Alice's four operations map the joint shared state between her and Bob to the four orthogonal states

$$\begin{aligned} |\phi\rangle_{23}^{(1)} &= \left[\sin\theta\cos\alpha\,|00\rangle + \sin\theta\cos\alpha\,|11\rangle\right. \\ &+ \left.\cos\theta\sin\alpha\,|10\rangle - \cos\theta\sin\alpha\,|01\rangle\right], \\ |\phi\rangle_{23}^{(2)} &= \left[\sin\theta\cos\alpha\,|00\rangle - \sin\theta\cos\alpha\,|11\rangle\right. \\ &- \left.\cos\theta\sin\alpha\,|10\rangle - \cos\theta\sin\alpha\,|01\rangle\right], \\ |\phi\rangle_{23}^{(3)} &= \left[\cos\theta\sin\alpha\,|00\rangle - \cos\theta\sin\alpha\,|11\rangle\right. \\ &+ \left.\sin\theta\cos\alpha\,|10\rangle + \sin\theta\cos\alpha\,|01\rangle\right], \\ |\phi\rangle_{23}^{(4)} &= \left[\cos\theta\sin\alpha\,|00\rangle + \cos\theta\sin\alpha\,|11\rangle\right. \\ &- \left.\sin\theta\cos\alpha\,|10\rangle + \sin\theta\cos\alpha\,|01\rangle\right]. \end{aligned} \tag{15}$$

All the four partially entangled states, Eq. (15), are orthogonal to each other for all the values of θ and α. Thus, in principle, Alice can produce four distinct messages for Bob by locally manipulating her qubit 2 and hence Bob can distinguish between her messages by performing appropriate joint measurements on the combined state of qubits 2 and 3. Although the dense coding capacity of the shared resource between Alice and Bob after controllers measurement is independent of α, in order to distinguish between the four distinct messages the receiver must know the value of α. For example at $\alpha = \pi/4$, Bob can first perform a two-qubit unitary operation $\sin\theta\left[|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|\right] + \cos\theta\left[|00\rangle\langle 10| + |01\rangle\langle 11| - |10\rangle\langle 00| - |11\rangle\langle 01|\right]$ and then perform a joint measurement in the Bell basis to decode the communicated message. In general, Bob can always distinguish between Alice's messages irrespective of the degree of entanglement of the initially shared partially entangled three-qubit state $|\chi\rangle_{123}^{(1)}$ between the three users and Charlie's measurement basis. Similarly for Charlie's measurement outcome $|x_-\rangle_1$ also, Bob can successfully decode the message and thus Alice can always send maximum number of bits to Bob, i.e., 2-bit using her one qubit. Hence, our set of states $|\chi\rangle_{123}^{(i)}$ can be used for perfect and deterministic information transfer irrespective of the amount of entanglement in the shared resource.

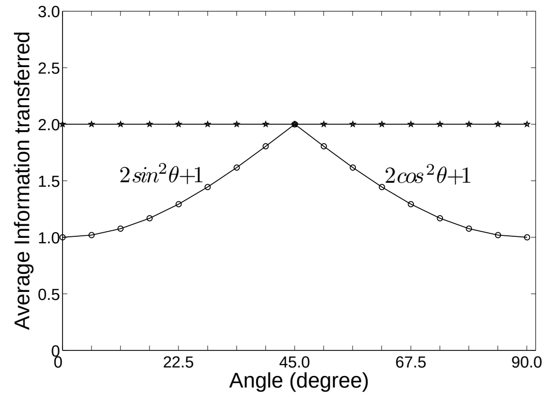

FIG. 3: Comparison of the average number of bits transferred via controlled dense coding using the GGHZ state (circles) versus the $|\chi\rangle_{123}^{(1)}$ state (stars) as a function of degree of entanglement (θ).

If the shared channel between Alice, Charlie and Bob is the MS state instead of the $|\chi\rangle_{123}^{(i)}$ state, then we can express the shared quantum channel in terms of Charlie's measurement basis as,

$$\begin{aligned} |MS\rangle_{123} &= \frac{|x_+\rangle_1}{\sqrt{2}}\left[\cos\alpha\,|00\rangle + \sin\alpha\cos\theta\,|10\rangle\right. \\ &+ \left.\sin\theta\sin\alpha\,|11\rangle\right] \\ &+ \frac{|x_-\rangle_1}{\sqrt{2}}\left[\sin\alpha\,|00\rangle - \cos\alpha\cos\theta\,|10\rangle\right. \\ &- \left.\sin\theta\cos\alpha\,|11\rangle\right]. \end{aligned} \tag{16}$$

It is clear that no measurement outcome of Charlie would allow Alice to produce four distinct messages for Bob. For example, if Charlie's measurement outcome is $|x_+\rangle_1$, then by performing local operations (I, σ_z^2, σ_x^2, $i\sigma_y^2$), Alice

can prepare four non-orthogonal states

$$\begin{aligned}
|\varphi\rangle_{23}^{1} &= \cos\alpha\,|00\rangle + \sin\alpha\cos\theta\,|10\rangle \\
&+ \sin\theta\sin\alpha\,|11\rangle\,, \\
|\varphi\rangle_{23}^{2} &= \cos\alpha\,|00\rangle - \sin\alpha\cos\theta\,|10\rangle \\
&- \sin\theta\sin\alpha\,|11\rangle\,, \\
|\varphi\rangle_{23}^{3} &= \sin\alpha\cos\theta\,|00\rangle + \cos\alpha\,|10\rangle \\
&+ \sin\theta\sin\alpha\,|01\rangle\,, \\
|\varphi\rangle_{23}^{4} &= \sin\alpha\cos\theta\,|00\rangle - \cos\alpha\,|10\rangle \\
&+ \sin\theta\sin\alpha\,|01\rangle\,.
\end{aligned} \tag{17}$$

Thus, by performing local unitary transformations, Alice can only produce two orthogonal states, i.e., can send only 1 bit information to Bob. Hence, using MS states as a quantum resource, a sender cannot transfer maximum information to a receiver. In addition, compared to the $|\chi\rangle_{123}^{(i)}$ states, if a GGHZ state is used as a shared resource then following the protocol outlined in [6], the average amount of information that can be transferred from a sender to a receiver would be $(1+2\sin^2\theta)$, for $0<\theta\leq\pi/4$, which depends on the degree of entanglement of the shared GGHZ state. A comparison of the protocols with the $|\chi\rangle_{123}^{(1)}$ and GGHZ states, based on the degree of entanglement of both the states, is illustrated in Fig. (3) and clearly demonstrates the advantages of using $|\chi\rangle_{123}^{(1)}$ states in comparison to the GGHZ states.

V. QUANTUM TELEPORTATION

In order to communicate an unknown quantum state $|\psi\rangle_1 = a\,|0\rangle_1 + b\,|1\rangle_1$ with Bob (Receiver), Alice (Sender) must share a partially entangled state $|\chi\rangle_{234}^{(i)}$, for e.g., $|\chi\rangle_{234}^{(1)} = \frac{1}{\sqrt{2}}\,[\sin\theta\,|000\rangle_{234} + \sin\theta\,|011\rangle_{234} + \cos\theta\,|110\rangle_{234} - \cos\theta\,|101\rangle_{234}]$, $0<\theta<\pi/4$, with Charlie (Controller) and Bob such that qubit 2 is with Charlie, qubit 3 is with Alice and qubit 4 is with Bob. In fact, Alice and Bob can have any of the qubits 3 or 4 in the shared entangled resource but qubit 2 is fixed for Charlie. The joint state of four qubits including the arbitrary qubit $|\psi\rangle_1$ and the shared quantum channel $|\chi\rangle_{234}^{(1)}$ can be written as

$$|\psi\rangle_{1234} \;=\; |\psi\rangle_1 \otimes |\chi\rangle_{234}^{(1)}\,. \tag{18}$$

A joint measurement, by Alice, in Bell basis on qubits 1 and 3 will project the joint state of Charlie and Bob's particles onto one of four possible states with equal probability of 1/4. For example, if Alice's measurement outcome is $|\phi\rangle_{13}^{+}$, then the joint state of rest of the qubits can be given as

$$\begin{aligned}
|\phi\rangle_{24} &= [a\sin\theta\,|00\rangle_{24} - a\cos\theta\,|11\rangle_{24} + b\sin\theta\,|01\rangle_{24} \\
&+ b\cos\theta\,|10\rangle_{24}]\,.
\end{aligned} \tag{19}$$

In order to teleport the unknown state to Bob successfully, Charlie measures his qubit 2 in the basis

$$\begin{aligned}
|x_+\rangle_2 &= \cos\alpha|0\rangle_2 + \sin\alpha|1\rangle_2, \\
|x_-\rangle_2 &= \sin\alpha|0\rangle_2 - \cos\alpha|1\rangle_2.
\end{aligned} \tag{20}$$

The joint state of qubits 2 and 4 can be re-expressed in terms of Charlie's measurement basis ($|x_+\rangle_2$, $|x_-\rangle_2$) as

$$\begin{aligned}
|\phi\rangle_{24} &= |x_+\rangle_2\,[(a\sin\theta\cos\alpha + b\cos\theta\sin\alpha)\,|0\rangle_4 \\
&+ (b\sin\theta\cos\alpha - a\cos\theta\sin\alpha)\,|1\rangle_4] \\
&+ |x_-\rangle_2\,[(a\sin\theta\sin\alpha - b\cos\theta\cos\alpha)\,|0\rangle_4 \\
&+ (a\cos\theta\cos\alpha + b\sin\theta\sin\alpha)\,|1\rangle_4]\,.
\end{aligned} \tag{21}$$

Thus, for Charlie's measurement in the basis ($|x_+\rangle_2$, $|x_-\rangle_2$), Bob's qubit will be projected onto one of the states $|\Psi\rangle_1$ or $|\Psi\rangle_2$ where

$$\begin{aligned}
|\Psi\rangle_1 &= [a\sin\theta\cos\alpha + b\cos\theta\sin\alpha]\,|0\rangle_4 \\
&+ [b\sin\theta\cos\alpha - a\cos\theta\sin\alpha]\,|1\rangle_4\,,
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
|\Psi\rangle_2 &= [a\sin\theta\sin\alpha - b\cos\theta\cos\alpha]\,|0\rangle_4 \\
&+ [a\cos\theta\cos\alpha + b\sin\theta\sin\alpha]\,|1\rangle_4\,.
\end{aligned} \tag{23}$$

Let us assume that Charlie's measurement outcome is $|x_+\rangle_2$, then in order to recover the teleported state successfully Bob needs to perform a single qubit unitary transformation

$$U_4 = \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}, \tag{24}$$

which will evolve the state of qubit 4 as

$$\begin{aligned}
U_4\,|\Psi\rangle_1 &= \frac{1}{\sqrt{N_1}}\,\big[(a\sin^2\theta\cos\alpha + b\sin\theta\cos\theta(\sin\alpha - \cos\alpha) \\
&+ a\cos^2\theta\sin\alpha)\,|0\rangle_4 \\
&+ (b\cos^2\theta\sin\alpha + a\sin\theta\cos\theta(\cos\alpha - \sin\alpha) \\
&+ b\sin^2\theta\cos\alpha)\,|1\rangle_4\big]\,,
\end{aligned} \tag{25}$$

where N_1 is the normalization constant given by

$$\begin{aligned}
N_1 &= \big[a\sin^2\theta\cos\alpha + b\sin\theta\cos\theta(\sin\alpha - \cos\alpha) \\
&+ a\cos^2\theta\sin\alpha\big]^2 \\
&+ \big[b\cos^2\theta\sin\alpha + a\sin\theta\cos\theta(\cos\alpha - \sin\alpha) \\
&+ b\sin^2\theta\cos\alpha\big]^2\,.
\end{aligned} \tag{26}$$

The square of the overlap between the input state $|\psi\rangle_1$ and the output state $U_4\,|\Psi\rangle_1$ is

$$\langle\psi_1|\,U_4|\Psi_1\rangle^2 = \frac{1}{N_1}\,\big[\sin^2\theta\cos\alpha + \cos^2\theta\sin\alpha\big]^2\,. \tag{27}$$

Similarly, for Charlie's measurement outcome $|x_-\rangle_2$, the required unitary transformation is

$$U_4' = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}, \tag{28}$$

and the overlap with the input state $|\psi_1\rangle$ is

$$\langle\psi_1|\, U_4'|\Psi_2\rangle^2 = \frac{1}{N_2}\left[\sin^2\theta\sin\alpha + \cos^2\theta\cos\alpha\right]^2. \quad (29)$$

Here

$$\begin{aligned} N_2 &= \big[a\sin^2\theta\sin\alpha + b\sin\theta\cos\theta(\sin\alpha - \cos\alpha) \\ &+ a\cos^2\theta\cos\alpha\big]^2 \\ &+ \big[b\cos^2\theta\cos\alpha + a\sin\theta\cos\theta(\cos\alpha - \sin\alpha) \\ &+ b\sin^2\theta\sin\alpha\big]^2. \end{aligned} \quad (30)$$

Hence, the average fidelity can be expressed as

$$\begin{aligned} \bar{F}_{+,-} &= P\left(x_+\right)\langle\psi_1|\, U_4|\Psi_1\rangle^2 + P\left(x_-\right)\langle\psi_1|\, U_4|\Psi_2\rangle^2 \\ &= \sin^4\theta + \cos^4\theta + 4\sin^2\theta\cos^2\theta\sin\alpha\cos\alpha. \end{aligned} \quad (31)$$

For $\alpha = \pi/4$, the average fidelity would be unity for any shared resource state. Moreover, the state of Bob's qubit will be the same as the state of Alice's qubit after performing the unitary transformation U_4 and teleportation will be successful. However, for all other α $(0 < \alpha < \pi/4)$ average fidelity, Eq. (31), depends on the controllers measurement basis, characterized by α and Bob can always recover the original state communicated by Alice in a similar fashion as discussed above. Thus, controlled teleportation is always successful irrespective of the degree of entanglement between the three qubits which may provide flexibility to the experimental set-ups by releasing the constraint of using a maximally entangled shared resource for faithful teleportation. Therefore, for a certain choice of Charlie's measurement basis, one can always obtain unit probability and fidelity of the teleportation process using our state $|\chi\rangle_{234}^{(i)}$ as an entangled resource. In comparison to our states, controlled teleportation using the three-qubit W state [8] as a shared entangled channel is probabilistic and the total probability of teleporting a single qubit depends on the unknown coefficients of the state to be teleported. The GGHZ states are also used to accomplish the same task using conclusive teleportation [14] and qubit assisted conclusive teleportation [15] but the total probability of the teleportation is not unity. Thus, our set of states results in improved and faithful deterministic teleportation of a single-qubit with standard measurements only and allow one to even consider partially entangled states for successful communication.

VI. CONCLUSION

To conclude, we have shown that the set of partially entangled states proposed in this article can be used faithfully for an optimal quantum information processing as compared to other partially entangled two and three-qubit states. The completion of quantum teleportation using our states as shared quantum channel with *unit probability and fidelity* is of particular importance because teleportation can be used as a secure intermediate process for many cryptographic schemes such as quantum secure direct communication. The quantum circuit illustrated in Fig. (1) shows that the physical realization of the $|\chi\rangle_{123}^{(i)}$ states from GGHZ states is simple enough to achieve but their entanglement properties are different from that of GGHZ states. The advantages of achieving perfect and optimal information transfer with the $|\chi\rangle_{123}^{(i)}$ states in various protocols also release the constraints in the experimental set-ups to prepare maximally entangled multiqubit states for secure and faithful quantum information processing. Interestingly, even though our states are nonlocally equivalent to the maximal slice (MS) states, they are locally inequivalent, leading to their enhanced efficiency in dense coding. Thus our study is of importance not only from the theoretical and experimental point of view but also sheds some light into the complex nature of multiqubit entanglement. Although the information transfer using the $|\chi\rangle_{123}^{(i)}$ states is always successful, however it would be interesting to study the usefulness of such an entangled resource under real experimental conditions when used for different information processing protocols. In an interesting recent work [25], a characterization of entanglement in terms of nonlocal semi-quantum games was provided. It would be of interest to study our results from the perspective of the above characterization. One could also design nonlocal quantum games based on LOSR (Local Operation and Shared Randomness) to show that the interconversion of GGHZ and MS to $|\chi\rangle$ states is not possible. Another topic of particular interest would be the generalization of our set to arbitrary number of qubits or to identify other partially entangled quantum channels with higher number of qubits for an optimal and successful quantum information processing.

Acknowledgements

A. K would like to acknowledge S. Ghose for introducing him to Svetlichny inequality and for other fruitful discussions.

[1] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).

[2] K. Bostrom and T. Felbinger, Phys. Rev. Lett. **89**,

187902 (2002).
[3] M. Zukowski, A. Zeilinger, M. A. Horne and A. Eckert, Phys. Rev. Lett. **71**, 4287 (1993).
[4] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[5] A. Karlsson and M. Bourennane, Phys. Rev. A **58**, 4394 (1998).
[6] J. C. Hao, C. F. Li and G. C. Guo, Phys. Rev. A **63**, 054301 (2001).
[7] T. Gao, Z. Naturforsch **59a**, 597 (2004).
[8] B. S. Shi and A. Tomita, Phys. Lett. A **296**, 161 (2002).
[9] Z. X. Man, Y. J. Xia and N. Ba An, Phys. Rev. A **75**, 052306 (2007).
[10] C. H. Bennett, H. J. Bernstein, S. Popescu and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
[11] D. M. Greenberger, M. A. Horne. A. Shimony and A. Zeilinger, A. J. Phys. **58**, 1131 (1990).
[12] H. A. Carteret and A. Sudbery, J. Phys. A: Math. Gen. **33**, 4981 (2000).
[13] W. Dur, G. Vidal and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
[14] T. Mor and P. Horodecki, e-print quant-ph/9906039.
[15] S. Bandyopadhyay, Phys. Rev. A **62**, 012308 (2000).
[16] P. Agrawal and A. K. Pati, Phys. Lett. A **305**, 12 (2002).
[17] P. Agrawal and A. Pati, Phys. Rev. A **74**, 062320 (2006).
[18] V. Coffman, J. Kundu and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).
[19] G. Svetlichny, Phys. Rev. D **35**, 3066 (1987).
[20] S. Ghose, N. Sinclair, S. Debnath, P. Rungta and R. Stock, Phys. Rev. Lett. **102**, 250404 (2009).
[21] A. Sawicki and M. Kus, arXiv: 1009.0293v1.
[22] A. Acin, D. Bruss, M. Lewenstein and A. Sanpera, Phys. Rev. Lett. **87**, 040401 (2001).
[23] N. Ganguly, S. Adhikari, A. S. Majumdar and J. Chatterjee, Phys. Rev. Lett. **107**, 270501 (2011); N. Ganguly and S. Adhikari, Phys. Rev. A **80**, 032331 (2009).
[24] D. Bruss, *et al.*, Phys. Rev. Lett. **93**, 210501 (2004).
[25] F. Buscemi, Phys. Rev. Lett. **108**, 200401 (2012).